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CONSERVATIVE GROUPS, INDICABILITY, AND A CONJECTURE OF HOWIE

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The class of locally p-indicable groups coincides with the class of $\mathbb{Z}_p$ conservative groups. Howie's conjecture is valid for locally p-indicable groups.

0. Introduction

In [6], J. Howie codified and generalized a conjecture that had been around for a number of years. Namely, if $G\langle t_1, \ldots, t_n\rangle$ is the free product of G with a free group freely generated by $t_1, \ldots, t_n$; if P is normally generated by $w_1, \ldots, w_m$, where the matrix $M=(e_j(w_i))$ of exponent sums of variables t_j in words w_i has independent rows; then the map $G \to G\langle t_1, \ldots, t_n\rangle/P$ should be injective. Howie proves this if G is locally indicable and references an old paper of Gerstenhaber and Rothaus [4] who prove it when G is finite (from which it follows easily if G is locally residually finite).

There are two essential ingredients in Howie's argument: a combinatorial argument about coverings (Section 5 below) and an old result of Adams [1], that the infinite cyclic group is $\mathbb{Z}$-conservative.

The notion of conservative group was apparently rediscovered by Strebel [13] (since he does not refer to Adams) and generalized. For any commutative ring R, he introduced a class $\mathscr{S}(R)$ of groups (we call them R-conservative below, Definition 2.1) which agrees with $\mathbb{Z}$-conservative groups if $R=\mathbb{Z}$. Strebel proved also that $\mathbb{Z}$-conservative groups are locally indicable.

In Sections 1 and 2 below we have gathered results on the various derived series of groups and on R-conservative groups, and have attempted to stabilize the terminology. We make no claim for originality there. Of particular interest is the Adams subgroup $A_p(G)$ of a group G (2.4 below). It is the smallest normal subgroup such that $G/A_p(G)$ is $\mathbb{Z}_p$-conservative. We show $A_p(G)$ is p-perfect, an argument that can be traced to Adams [1].

In Section 3 we introduce the class of locally p-indicable groups (l.p.i.) (3.1). If $p=\infty$, this is the classical notion of locally indicable groups due to Higman [5]. We prove a criterion (3.4) for a group to be l.p.i. in terms of graph products [12] of l.p.i. vertex groups whose edge groups satisfy a uniformity condition (3.3). It follows that Baumslag's group $\langle a,b \mid a=[a^b,a]\rangle$, the Baumslag–Solitar groups [2] $\langle a,b \mid a^n = ba^m b^{-1}, m\neq 0 \neq n\rangle$, and groups including torus knots $\langle a,b \mid a^m=b^n, m\neq 0\neq n\rangle$ are locally indicable (3.5).

In Section 4 we state the main theorem (4.1) that locally p-indicable groups coincide with $\mathbb{Z}$-conservative groups. The proof occupies Sections 4–6.[1]

In Section 7 we give the application to Howie's conjecture. In the notation of the first paragraph of this introduction, it states that if the rows of M, reduced (mod p), are linearly independent, then the kernel of $G \to G\langle t_1,\dots,t_n\rangle/P$ is contained in $A_p(G)$. In particular, the kernel is trivial in G is locally p-indicable.

This work was done while on sabbatical leave from the University of Utah.

1. p-derived series

1.1. The classical derived series $\{D^\alpha G\}$ of a group G is defined by $D^0G=G$ and $D^{\alpha+1}G$ is the commutator subgroup of $D^\alpha G$. If α is a limit ordinal, $D^\alpha G=\bigcap_{\beta<\alpha} D^\beta G$. Since the successive quotients $D^\alpha G/D^{\alpha+1}G$ can have arbitrary torsion, this series is of little use in the sequel.

1.2. If p is a prime number and G a group, one lets D_pG be the subgroup of G generated by all commutators $[x,y]=xyx^{-1}y^{-1}$ and p-th powers x^p, $x,y\in G$. Thus D_pG is smallest normal subgroup of G whose factor group G/D_pG is a $\mathbb{Z}_p$ $(=\mathbb{Z}/p\mathbb{Z})$ vector space. one defines $\{D_p^\alpha G\}$ inductively by the rule $D_p^{\alpha+1}G=D_p(D_p^\alpha G)$ and by intersections for limit ordinals $\alpha: D_p^\alpha G=\bigcap_{\beta<\alpha} D_p^\beta G$.

1.3. If A is an abelian group, let $\tau(A)$ be the torsion subgroup of A. For a general group G, let G_{ab} be G/DG and set $D_\infty G=$

$$\operatorname{Ker}(G \twoheadrightarrow G_{\mathrm{ab}}/\tau(G_{\mathrm{ab}})).$$

Thus $D_\infty G$ is the smallest normal subgroup of G whose factor group $G/D_\infty G\cong G_{\mathrm{ab}}/\tau(G_{\mathrm{ab}})$ is torsion free abelian. One defines $\{D_\infty^\alpha G\}$ for ordinals α as before.

1.4. For cardinality reasons, each of the series $\{D^\alpha G\}$, $\{D_p^\alpha G\}$, $\{D_\infty^\alpha G\}$ stabilizes with normal subgroups $\Pi(G)$, $\Pi_p(G)$, $\Pi_\infty(G)$ respectively. We have $\Pi(G)/D\Pi(G)=\Pi_p(G)/D_p\Pi_p(G)=\Pi_\infty(G)/D_\infty\Pi_\infty(G)=\{1\}$, motivating the following definition:

Definition. A group G is *perfect* or *p-perfect* (p prime or ∞) if G/DG respectively G/D_pG (p prime or ∞) is trivial.

[1] Howie informed me that he and H. Schneebeli have independently given a proof of Theorem 4.1.

1.5. For any group G, $\Pi(G)$ (respectively $\Pi_p(G)$, p prime or ∞) is the maximal perfect (respectively p-perfect) subgroup of G. From now on we adopt the convention that p is a prime number of ∞. The ring $\mathbb{Z}_p$ denotes $\mathbb{Z}/p\mathbb{Z}$ if p is prime and $\mathbb{Z}_\infty = \mathbb{Z}$.

1.6. Example. Let $G = \langle x, y \mid yxy^{-1} = x^{-1} \rangle$, the Klein bottle group. Then $DG = \langle x^2 \rangle$ but $D_\infty G = \langle x \rangle$. Also $\Pi(G) = \Pi_p(G) = \{1\}$ $\forall p \leq \infty$.

2. p-conservative groups

2.1. If R is a commutative ring with unit and RG is the group ring of the group G, one says G is *R-conservative* if a homomorphism $f : P \to Q$ of fg projective RG modules with $1 \otimes_{RG} f : R \otimes_{RG} P \to R \otimes_{RG} Q$ injective is itself injective. It is immediate that the condition need only be checked on fg free RG modules. This notion is due to Adams [1] who used it to establish special cases of the Whitehead Conjecture. It was rediscovered by Strebel [13], who called such a group a $\mathcal{S}(R)$-group, and who established again many of the closure properties of the class of R-conservative groups as well as new properties. For historical reasons as well as pyschological we adopt Adams' terminology.

2.2. The closure properties of the class $\mathcal{S}(R)$ of R-conservative groups may be summarized in:

Theorem [1, 13]. (1) *The class $\mathcal{S}(R)$ is closed under filtering direct limits, inverse limits and arbitrary cartesian products.*

(2) *Any subgroup of an R-conservative group is R-conservative.*

(3) *If G has a transfinite subnormal series $\{G_\alpha, \alpha \leq \sigma\}$, $G_0 = G$, $G_{\alpha+1} \lhd G_\alpha$, $G_\sigma = \{1\}$ with $G_\alpha / G_{\alpha+1}$ R-conservative for all α, then G is R-conservative.*

(4) *If $R = \mathbb{Z}_p$ (p prime or ∞) then the cyclic group $\mathbb{Z}_p$ is $\mathbb{Z}_p$-conservative.*

Remark. An immediate consequence of (1) and (4) is that any $\mathbb{Z}_p$-vector space is $\mathbb{Z}_p$-conservative and any torsion free abelian group is $\mathbb{Z}$-conservative.

2.3. Proposition. *If G is a group such that $\Pi_p(G) = \{1\}$ (p prime or ∞) then G is $\mathbb{Z}_p$-conservative.*

Proof. This is immediate from 2.2 (3) and the remark.

This result is equivalent to saying that a group containing no nontrivial p-perfect subgroups is $\mathbb{Z}_p$-conservative.

2.4. Proposition. *Let G be a group and define $A_p(G) = \bigcap \{N : N \lhd G, G/N$ $\mathbb{Z}_p$-conservative$\}$. Then $G/A_p(G)$ is $\mathbb{Z}_p$-conservative and $A_p(G)$ is p-perfect.*

Proof. $G/A_p(G)$ is a subgroup of $\prod\{G/N: N\lhd /N\ \mathbb{Z}_p\text{-conservative}\}$. Thus the first conclusion follows from 2.2 (1) and (2). As for the second, consider the short exact sequence of groups $1\to A_p(G)/D_pA_p(G)\to G/D_pA_p(G)\to G/A_p(G)\to 1$. By the remark following 2.2, $A_p(G)/D_pA_p(G)$ is $\mathbb{Z}_p$-conservative, and $G/A_p(G)$ is $\mathbb{Z}_p$-conservative. The result follows from 2.2 (3).

We shall $A_p(G)$ the Adams p-subgroup of the group G. Observe that $A_p(G)\le \Pi_p(G)$ since $G/\Pi_p(G)$ has no nontrivial p-perfect subgroups, hence is $\mathbb{Z}_p$-conservative by 2.3.

2.5. The following result summarizes the connection between the derived series $\{D^\alpha G\}$ and $\{D^\alpha_\infty G\}$ for an important class of groups.

Theorem [13, 3, 1]. *Let G be the fundamental group of a connected 2-complex X such that $H_2(X;\mathbb{Z})=0$ and $H_1(X;\mathbb{Z})$ is torsion free.*
(1) *For all ordinals α, $D^\alpha G=D^\alpha_\infty G$, so $\Pi(G)=\Pi_\infty(G)$.*
(2) *$H_i(\Pi(G),\mathbb{Z})=0$ for $i=1,2$ and $H_i(A_\infty(G),\mathbb{Z})=0$ for $i=1,2$.*
(3) *$G/\Pi(G)$ and $G/A_\infty(G)$ both have* $\mathrm{cd}_{\mathbb{Z}}\le 2$.

The statements for $\Pi(G)$ are due to Strebel [13] as well as (3), whereas (2) for $A_\infty(G)$ is due to Cohen [3].

2.6. There is an analogous result for $\{D^\alpha_p G\}$, which is proved by identical methods:

Theorem. *Let G be the fundamental group of a connected 2-complex X with $H_2(X;\mathbb{Z}_p)=0$ $(p\le\infty)$. Then*
(1) *$H_i(\Pi_p(G);\mathbb{Z}_p)=H_i(A_p(G);\mathbb{Z}_p)=0$ for $i=1,2$.*
(27 *$G/\Pi_p(G)$ and $G/A_p(G)$ have* $\mathrm{cd}_{\mathbb{Z}_p}\le 2$.

2.7. As an example of the preceding ideas suppose G is a $\mathbb{Z}$-conservative group and a Magnus group, so G has a presentation $\mathscr{P}$ with deficiency equal to the rank of $H_1(G)$. We claim G has a 2-dimensional $K(G,1)$. To see this, let X be the 2-complex constructed from the presentation $\mathscr{P}$: one circle for each generator, one 2-cell for each relator. An Euler characteristic computation shows rank $H_2(X;\mathbb{Z})=0$, whence $H_2(X;\mathbb{Z})=0$. Since G is $\mathbb{Z}$ conservative, one deduces $H_2(\tilde{X};\mathbb{Z})=0$, where $\tilde{X}$ is the universal cover of X. Thus X is aspherical, $X\simeq K(G,1)$.

3. Locally p-indicable groups

3.1. Definition. A group G is p-indicable $(p\le\infty)$ if either $G=\{1\}$ or G admits a homomorphism onto $\mathbb{Z}_p$. The group G is *locally p-indicable* (abbreviated l.p.i. in the sequel) if every fg subgroup of G is p-indicable.

The notion has its origins in G. Higman's thesis [5], where he defined a group G to be indicable if it mapped nontrivially to $\mathbb{Q}$ and thorougly indicable if every nontrivial subgroup was indicable. This latter notion is quite restrictive, since it implies $\Pi_\infty(G)=1$, thus does not allow the possibility of nontrivial ∞-perfect subgroups. The notion that has been generally accepted is that of *locally indicable groups* which coincides with our locally-∞-indicable (l.∞.i) groups. These groups may have ∞-perfect subgroups, but all such are infinitely generated.

It is not difficult to show directly that the class of l.p.i. groups is closed under the operations of (1), (2), (3) in 2.2; namely filtering directed limits, inverse limits, products, subgroups, and transfinite extensions; and of course $\mathbb{Z}_p$ is l.p.i. We omit the details since a much stronger result is true: G is l.p.i. iff G is $\mathbb{Z}_p$-conservative. We offer here another method of producing l.p.i. groups.

3.2. Recall [12] a graph of groups consists of an oriented (connected) graph Γ; for each vertex v and edge e of Γ groups G_v, G_e, and *injections* $G_e \to G_{\partial_0 e}$, $G_e \to G_{\partial_1 e}$, where $\partial_0 e$ is the initial vertex of e and $\partial_1 e$ is the terminal vertex (which may be the same vertex)

$$\partial_0 e \xrightarrow[e]{} \partial_1 e.$$

The graph product or fundamental group G_Γ[12, 10] is defined. In effect, one chooses $X_v = K(G_v, 1)$ and $X_e = K(G_e, 1)$ and glues $X_e \times I$ to $X_{\partial_0 e}$ and $X_{\partial_1 e}$ at the ends of the interval via the given maps $G_e \to G_{\partial_i e}$. In the resulting space X, one chooses a base point, and sets $\pi_1(X) = G_\Gamma$. The key result we need is the subgroup theorem, which states that any subgroup $H \le G_\Gamma$ is itself a graph product, where edge groups (respectively vertex groups) are of the form $H \cap xG_e x^{-1}$ (respectively $H \cap xG_v x^{-1}$) for certain $x \in G_\Gamma$.

3.3. Definition. A group G is *uniformly p rank* 1 if for any nontrivial subgroup $S \le G$, $H_1(S; k)$ is of k-dimension 1. Here k is any field of characteristic p (if $p=\infty$, this means "characteristic 0"). As examples, $\mathbb{Z}$ is uniformly p rank 1 for all p; $\mathbb{Z}_p$ is of uniform p rank 1; and any subgroup of $\mathbb{Q}$ is of uniform ∞-rank 1. A more exotic example is $G = \langle a, b \mid a = [b, a] \rangle$. It is not difficult to show that G is of uniform ∞ rank 1.

3.4. Theorem . *Let $p \le \infty$ and let (Γ, G_v, G_e) be a graph of groups, where the vertex groups G_v are l.p.i. and edge groups G_e are uniformly p rank* 1. *Then the graph product G is l.p.i.*

Proof. Let $1 \ne H \le G_\Gamma$ where H is fg. Then H is a graph product by the subgroup theorem. Let the graph be Γ_1, vertex and edge groups H_v, H_e for vertices v, e of Γ_1.

Since H is fg, we may assume, without loss of generality, that Γ_1 is finite. If T_1 *is a maximal tree of* Γ_1, any edge e of $\Gamma_1 \setminus T_1$ corresponds to an HNN extension, hence a nontrivial homomorphism $H \twoheadrightarrow \mathbb{Z} \twoheadrightarrow \mathbb{Z}_p$. Thus we are done unless $T_1 = \Gamma_1$, which we assume from now on is the case.

We may exhaust T_1, beginning with a vertex v, adding one edge at a time, in such a way that at each stage the partial graph is connected, hence a tree. In this way, by a secondary induction on a number of edges of T_1, we are reduced to showing the theorem in the case Γ is one edge with two distinct vertices, v_0 and v_1:

$$v_0 \xrightarrow{e} v_1.$$

Assume then that H_{v_0} and H_{v_1} are l.p.i. and H_e is of uniform p-rank 1; $H = H_{v_0} *_{H_e} H_{v_1}$. Let k be a field of characteristic p (i.e. characteristic 0 if $p=\infty$) and compute $H_1(H;k)$. By the Mayer–Vietoris Theorem, there is an exact sequence

$$H_1(H_e;k) \to H_1(H_{v_0};k) \oplus H_1(H_{v_1};k) \to H_1(H;k) \to 0. \qquad (*)$$

If H_{v_0} is trivial, then $H_e = \{1\}$, so $H_1(H;k) = H_1(H_{v_1};k) \neq 0$, since $H_{v_1} \neq \{1\}$ (otherwise $H=1$) and H_{v_1} is l.p.i. Since H is fg, this implies the existence of a homomorphism $H \twoheadrightarrow \mathbb{Z}_p$. The argument if $H_{v_1} = \{1\}$ is similar.

If neither H_{v_0} nor H_{v_1} is trivial, then $\dim_k H_1(H_{v_i};k) \geq 1$. Since $\dim_k H_1(H_e;k) \leq 1$ (H_e is uniformly p-rank 1) we deduce from the exact sequence $(*)$ that $\dim_k H_1(H;k) \geq 1$. Again this implies the existence of a map $H \twoheadrightarrow \mathbb{Z}_p$ since H is fg. This completes the proof of 3.4.

3.5. Examples. Let $B = \langle a, b \mid a = [a^b, a] \rangle$ where $a^b = bab^{-1}$, $[x,y] = xyx^{-1}y^{-1}$. We claim B is l.p.i. for all p. But B retracts onto $\langle b \rangle$ by $a \to 1$ and the proof of the Freiheitssatz [8] shows that $B = \langle a_0, a_1, b \mid a_0 = [a_1, a_0], ba_0b^{-1} = a_1 \rangle$. Thus B is a graph product with vertex group $\langle a_0, a_1 \mid a_0 = [a_1, a_0] \rangle = G$ and edge group $\mathbb{Z}$.

We have already remarked that G is l.p.i. and $\mathbb{Z}$ is uniformly p rank 1. Thus B is l.p.i.

The example is of interest since the commutator subgroup DB is perfect.

Likewise the Baumslag–Solitar groups $\langle a, b \mid ba^m b^{-1} = a^n, m \neq 0 \neq n \rangle$ and the groups $\langle a, b \mid a^m = b^n, m \neq 0 \neq n \rangle$ are l.p.i.

3.6. Remark. One may ask whether Example 3.5 is indicative of a more general phenomenon: whether torsion free 1-relator groups are $\mathbb{Z}$-conservative. Lydon's identity theorem gives some evidence for this in the following sense. Let F be free on $a_1, \dots, a_m$ and let $1 \neq R \subset F$, $R \notin DF$, R not a proper power in F. Let $G = \langle a_1, \dots, a_n \mid R \rangle$. Since the deficiency of the presentation for G is the rank of $H_1(G)$, if G were conservative it would, by 2.6, have a 2-dimensional $K(G,1)$. This is indeed the case, by Lydon's theorem [L].[2]

[2] J. Howie informed me that S.D. Brodskii established that torsion free 1-relator groups are locally indicable (Uspekhi Mat. Nauk 35, 4 (1980)).

3.7. Remark. There is an amalgam $F *_H G$ which is perfect (even acyclic), where F and G are free of rank 2 and H is free of rank 4. For let $H=\langle a,b,c,d\rangle$. There is an injection of H into F which maps $\langle a,b\rangle$ homologically onto F (i.e. an $\cong$ on H_1) and takes c,d into DF. Similarly there is an injection of H into G which maps $\langle c,d\rangle$ homologically onto G and takes a,b into DG. The Mayer–Vietoris Theorem shows that $F *_H G$ so constructed is acyclic. Thus $F *_H G$ is certainly not l.p.i. I learned of this example from J. Stallings.

4. Statement of the main result

4.1. Theorem. *The class of locally p-indicable groups coincides with the class of* $\mathbb{Z}_p$*-conservative groups* ($p \le \infty$).

The fact that $\mathbb{Z}$-conservative groups are locally indicable is due to Strebel [13]. The generalization, that $\mathbb{Z}_p$-conservative implies l.p.i., involves such slight modifications of Strebel's argument that it is not worth repeating here.

The proof of the converse, that locally p-indicable implies $\mathbb{Z}_p$-conservative, involves a geometric tower argument based on ideas of Howie [6]. The argument will be given in succeeding sections. Here we set up the algebraic machinery used in the last part of the proof.

4.2. Suppose G is a group, R is a fixed commutative ring with unit, and RG is the group ring. A *pattern* (over R) is a 5-tuple $(G, f: P \to Q, x)$ where f is a homomorphism of fg free RG modules P, Q such that

$$1 \otimes_{RG} f : R \otimes_{RG} P \to R \otimes_{RG} Q$$

is injective and such that $x \in P$ with $f(x)=0$. Observe that G is R-conservative iff for each pattern $(G, f: P \to Q, x)$ one has $x=0$.

A morphism of patterns

$$(G_1, f_1 : P_1 \to Q_1, x_1) \xrightarrow{(\phi,\alpha,\beta)} (G, f: P \to Q, x)$$

consists of a homomorphism of groups $\phi: G_1 \to G$, and homomorphisms of RG-modules

$$\alpha : RG \otimes_{RG_1} P_1 \to P, \qquad \beta : RG \otimes_{RG_1} Q_1 \to Q$$

such that the following diagram commutes

$$\begin{array}{ccc} RG\otimes_{RG_1} P_1 & \xrightarrow{\alpha} & P \\ \Big\downarrow{\scriptstyle 1\otimes f_1} & & \Big\downarrow{\scriptstyle f} \\ RG\otimes_{RG_1} Q_1 & \xrightarrow{\beta} & Q \end{array}$$

and such that $\alpha(1 \otimes x_1) = x$. Here RG is a right RG_1 module via ϕ.

It is a straightforward matter to check that composition of morphisms of patterns is well defined and that the identity is a morphism, so one has a category of patterns.

4.3. Lemma. *If*

$$(G_1, f_1 : P_1 \to Q_1, x_1) \xrightarrow{(\phi, \alpha, \beta)} (G, f : P \to Q, x)$$

is a morphism of patterns and $\phi : G_1 \to G$ is trivial, then $x = 0$.

Proof. $\phi : G_1 \to G$ factors through 1, so one has a commutative diagram

$$\begin{array}{ccccc} P_1 & \xrightarrow{y \to 1 \otimes y} & RG \otimes_{RG_1} P_1 & \xrightarrow{\alpha} & P \\ \downarrow & & \downarrow \cong & & \\ R \otimes_{RG_1} P_1 & \longrightarrow & RG \otimes_R (R \otimes_{RG_1} P_1) & & \end{array}$$

The image of x_1 under $P_1 \to RG \otimes_{RG_1} P_1 \xrightarrow{\alpha} P$ is x by hypothesis. But the image of x_1 under $P_1 \to R \otimes_{RG_1} P_1$ is 0, since $f_1(x_1) = 0$ and $1 \otimes f_1 : R \otimes_{RG_1} P_1 \otimes_{RG_1} Q_1$ is injective. A diagram chase completes the proof.

4.4. We can now recast 4.1 in the form in which it will be proved.

Theorem. *Let $R = \mathbb{Z}_p$ $(p \leq \infty)$ and let $(G, f : P \to Q, x)$ be any pattern (over $\mathbb{Z}_p$). Then there is a morphism of patterns*

$$(G_1, f_1 : P_1 \to Q_1, x_1) \xrightarrow{(\phi, \alpha, \beta)} (G, f : P \to Q, x)$$

with G_1 fg and p-perfect.

The proof will occupy succeeding sections.

4.5. Corollary. *Let G be locally p-indicable. Then G is $\mathbb{Z}_p$-conservative.*

Proof. Let $f : P \to Q$ be a homomorphism of fg free $\mathbb{Z}_pG$ modules such that $1 \otimes f : \mathbb{Z}_p \otimes_{\mathbb{Z}_pG} P \to \mathbb{Z}_p \otimes_{\mathbb{Z}_pG} Q$ is injective. Let $x \in P$ with $f(x) = 0$, and consider the pattern $(G, f : P \to Q, x)$. By Theorem 4.4 there is a morphism of patterns

$$(G_1, f_1 : P_1 \to Q_1, x_1) \xrightarrow{(\phi, \alpha, \beta)} (G, f : P \to Q, x)$$

with G_1 fg and p-perfect. Since G is l.p.i., it follows that $\phi(G_1) = \{1\}$. Lemma 4.3 implies $x = 0$. The proof of 4.5 is complete, assuming Theorem 4.4.

5. Towers and homology

5.1. A *tower* is a map $K \xrightarrow{g} L$ of connected CW complexes such that g is a finite composition of inclusions of connected subcomplexes and of connected covering maps. If the inclusions all belong to some class $\mathscr{A}$ and the coverings to some class $\mathscr{B}$, g is called an $(\mathscr{A}, \mathscr{B})$ tower.

5.2. An $(\mathscr{A}, \mathscr{B})$-tower lifting of a map $X \xrightarrow{f} L$ of connected CW complexes is a commutative triangle

$$\begin{array}{ccc} & & K \\ & \overset{f'}{\nearrow} & \\ X & & \downarrow g \\ & \underset{f}{\searrow} & \\ & & L \end{array}$$

where g is an $(\mathscr{A}, \mathscr{B})$-tower. The lifting f' is maximal if the only $(\mathscr{A}, \mathscr{B})$-tower lifting of F' is the trivial one,

$$\begin{array}{ccc} & & K \\ & \overset{f'}{\nearrow} & \\ X & & \downarrow 1 \\ & \underset{f'}{\searrow} & \\ & & L \end{array}$$

5.3. A map $K \xrightarrow{f} L$ of CW complexes is *combinatorial* if it maps each open cell of K homeomorphically onto a cell of L. Combinatorial maps possess images in the CW-category. Every inclusion of a subcomplex is combinatorial. If $p: X \to Y$ is a covering map with Y a CW complex, then X can be given a (unique) CW structure such that p is combinatorial.

5.4. Lemma [6]. *Let $\mathscr{A}$ be a class of inclusion maps of* CW *complexes including inclusions of finite connected subcomplexes, and let $\mathscr{B}$ be a class of coverings. Let X, L be connected* CW *complexes with X finite. Then any combinatorial map $f: X \to L$ has a maximal $(\mathscr{A}, \mathscr{B})$-tower lifting.* □

5.5. Proposition. *Let L be a* 1*-connected* CW *complex and let $z \in H_n(L; \mathbb{Z}_p)$. There is a finite* 1*-connected* CW *complex X, a class $\xi \in H_n(X; \mathbb{Z}_p)$, and a combinatorial map $f: X \to L$ such that $f_*(\xi) = z$.*

Proof. By considering closures of carriers of a representative chain of z, we see there is a finite connected subcomplex L_1 of L, $z_1 \in H_n(L_1; \mathbb{Z}_p)$, such that $i_*(z_1) = z$ where i is the inclusion. Let $g_1, \ldots, g_r$ be generators for $\pi_1(L_1; v)$, v any 0-cell. Since

g_i is null homotopic in L, by [H] there is a combinatorial map $Y_i \xrightarrow{h_i} L$ such that

(1) Y_i is a planar 1-connected 2-complex,

(2) $h_i(Y_i^{(1)}) \subset L_1$ and the image

$$[S^1, Y_i^{(1)}] \xrightarrow{h_{i,*}} [S^1, L_1]$$

contains the conjugacy class of g_i.

Form the adjunction space X by attaching Y_i to L_1 via $h_i \mid Y_i^{(1)}$ for all i. Extend the inclusion $L_1 \xrightarrow{i} L$ to a map $f: X \to L$ by $f \mid Y_i = h_i$. Clearly f is combinatorial. Since $Y_i^{(1)}$ is contractible in Y_i and since the g_i generate $\pi_1(L_1; v)$, the van Kampen theorem [9] implies X is 1-connected. If $j: L_1 \to X$ is the inclusion, define $\xi = j_*(z_1) \in H_n(X; \mathbb{Z}_p)$. Since $f \circ j = i$, we have $f_*(\xi) = z$, which completes the proof.

5.6. Let G be a group and P and Q fg free $\mathbb{Z}_pG$ modules. let $f: P \to Q$ be a $\mathbb{Z}_pG$-homomorphism.

Proposition. *There is a* CW *pair* (L, K) *such that* L *and* K *are connected with base point in* K *and*

(1) $L^{(2)} = K$,

(2) $L \setminus K$ *consists of cells in only two adjacent dimensions* n *and* $n+1$, $n > 2$,

(3) $\pi_1(K) = \pi_1(L) = G$, *and*

(4) *if* $(\tilde{L}, \tilde{K})$ *is the universal cover pair, then the connecting homomorphism*

$$d_{n+1}: C_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}_p) \to C_n(\tilde{L}, \tilde{K}; \mathbb{Z}_p)$$

is identified with $f: P \to Q$.

Proof. Pick bases $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ for P and Q respectively, so we may identify P and Q with $(\mathbb{Z}_pG)^r$ and $(\mathbb{Z}_pG)^s$ (row vectors) respectively. Let

$$f(p_i) = \sum_{j=1}^{s} a_{ij} q_j, \quad a_{ij} \in \mathbb{Z}_pG,$$

and choose $\alpha_{ji} \in \mathbb{Z}G$ with $\alpha_{ij} \to a_{ij}$ under $\mathbb{Z}G \twoheadrightarrow \mathbb{Z}_pG$. Then f can be viewed as reduction (mod p) of the homomorphism $(\mathbb{Z}G)^r \to (\mathbb{Z}G)^s$ with matrix (α_{ij}).

Let K be a pointed connected 2-complex with $\pi_1(K) = G$. Let $n > 2$. Let $L_1 = K \vee S_1^n \vee \cdots \vee S_s^n$, where S_i^n is a copy of S^n, all attached at the base point of K. Then $H_n(\tilde{L}_1, \tilde{K}; \mathbb{Z}) \cong (\mathbb{Z}G)^s$. By the relative Hurewicz theorem, $\pi_n(\tilde{L}_1, \tilde{K}) = H_n(\tilde{L}_1, \tilde{K}; \mathbb{Z}) = (\mathbb{Z}G)^s$. Thus given $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{is}) = r_i \in (\mathbb{Z}G)^s$, we may attach an $(n+1)$-cell e_i^{n+1} to L_1 by the class of map $\partial e_i^{n+1} \to L_1$ represented by $r_i \in \pi_n(L, K) = \pi_n(\tilde{L}, \tilde{K})$. Thus we get

$$L =: L_1 \vee_{r_1} e_1^{n+1} \vee \cdots \vee_{r_n} e_r^{n+1}.$$

Since $C_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}) = H_{n+1}(\tilde{L}, \tilde{L}_1; \mathbb{Z}) = (\mathbb{Z}G)^r$ and $C_n(\tilde{L}, \tilde{K}; \mathbb{Z}) = H_n(\tilde{L}_1, \tilde{K}; \mathbb{Z}) = (\mathbb{Z}G)^s$, the construction shows that

$$d_{n+1}: C_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}) \to C_n(\tilde{L}, \tilde{K}; \mathbb{Z})$$

has matrix (α_{ij}). Thus the reduction (mod p)

$$d_{n+1}: C_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}_p) \to C_n(\tilde{L}, \tilde{K}; \mathbb{Z}_p)$$

has matrix (a_{ij}) and has been identified with $f: P \to Q$ as homomorphisms of $\mathbb{Z}_p G$ modules. This completes the proof of 5.6.

6. Proof of Theorem 4.4.

6.1. Let $(G, f: P \to Q, x)$ be a pattern over $\mathbb{Z}_p$. Recall $f: P \to Q$ is a homomorphism of fg free $\mathbb{Z}_p G$ modules, $x \in \ker f$, and $1 \otimes_{\mathbb{Z}_p G} f$ is injective. By Proposition 5.6, there is a relative CW pair (L, K), $L^{(2)} = K$, $\pi_1(L) = \pi_1(K) = G$, $L \backslash K$ a union of n and $(n+1)$-cells $(n > 2)$, such that $d_{n+1}: C_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}_p) \to C_n(\tilde{L}, \tilde{K}; \mathbb{Z}_p)$ is identified with $f: P \to Q$. Since $1 \otimes_{\mathbb{Z}_p G} d_{n+1}$ is the chain boundary

$$C_{n+1}(L, K; \mathbb{Z}_p) \to C_n(L, K; \mathbb{Z}_p)$$

we see that $x \in H_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}_p)$ and $H_{n+1}(L, K; \mathbb{Z}_p) = 0$.

Conversely, if (L, K) is a connected CW pair satisfying conditions (1), (2), and (3) of Proposition 5.6, and if $H_{n+1}(L, K; \mathbb{Z}_p) = 0$, then any $x \in H_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}_p)$ determines the pattern

$$(G, d_{n+1}: C_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}_p) \to C_n(\tilde{L}, \tilde{K}; \mathbb{Z}_p), x)$$

with $G = \pi_1(L)$.

6.2. Let $(L_1, K_1) \xrightarrow{\phi} (L, K)$ be a combinatorial map of pointed connected CW pairs with $L_1^{(2)} = K_1$, $L^{(2)} = K$ and $L_1 \backslash K_1$ and $L \backslash K$ consisting of only n and $(n+1)$-cells $(n > 2)$. Assume $H_{n+1}(L_1, K_1; \mathbb{Z}_p) = H_{n+1}(L, K; \mathbb{Z}_p) = 0$. Let $x_1 \in H_{n+1}(\tilde{L}_1, \tilde{K}_1; \mathbb{Z}_p)$.

$$\begin{array}{ccc}
(\tilde{L}_1, \tilde{K}_1) & \xrightarrow{\tilde{\phi}} & (\tilde{L}, \tilde{K}) \\
\downarrow & & \downarrow \\
(L_1, K_1) & \xrightarrow{\phi} & (L, K)
\end{array}$$

Choice of base points in the universal cover pairs determines a map $\tilde{\phi}: (\tilde{L}_1, \tilde{K}_1) \to (\tilde{L}, \tilde{K})$ covering ϕ. Furthermore the situation is equivariant. If $h \in G_1 = \pi_1(L_1)$, then $\tilde{\phi} \circ \tau_h = \tau_{\phi_*(h)} \circ \tilde{\phi}$ when $\phi_*: G_1 \to G = \pi_1 L$ is induced by ϕ and τ_h denotes the deck transformation of $(\tilde{L}_1, \tilde{K}_1)$. Let $x = \tilde{f}_*(x_1) \in H_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}_p)$.

Lemma. *$\tilde{\phi}$ determines a morphism of patterns*

$$\left(G_1, C_{n+1}(\tilde{L}_1, \tilde{K}_1; \mathbb{Z}_p) \xrightarrow{d_{n+1}} C_n(\tilde{L}_1, \tilde{K}; \mathbb{Z}_p), x_1\right)$$

$$\to \left(G, C_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}_p) \xrightarrow{d_{n+1}} C_n(\tilde{L}, \tilde{K}; \mathbb{Z}_p), x\right).$$

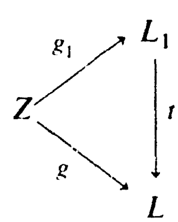
This follows since $\tilde{\phi}$ induces a chain map $C_1(\tilde{L}_1, \tilde{K}_1) \to C_1(\tilde{L}, \tilde{K})$.

6.3. Let $(G, f: P \to Q, x)$ be a pattern over $\mathbb{Z}_p$ and let (L, K) be the relative CW pair constructed in 6.1 from this pattern, so $x \in H_{n+1}(\tilde{L}, \tilde{K}; \mathbb{Z}_p) = H_{n+1}(\tilde{L}; \mathbb{Z}_p)$ (since $n > 2$ and $L^{(2)} = K$). By Proposition 5.5 there is a finite 1-connected CW complex Z, a class $z \in H_{n+1}(Z; \mathbb{Z}_p)$, and a combinatorial map $\tilde{g}: Z \to \tilde{L}$ such that $\tilde{g}_*(z) = x$. Let $g: Z \to L$ be the composite

$$Z \xrightarrow{\tilde{g}} \tilde{L} \longrightarrow L.$$

Thus g is a combinatorial map.

We may apply Lemma 5.4 with $\mathcal{A}$ the class of inclusion maps of finite subcomplexes and $\mathcal{B}$ the class of connected regular coverings with groups of covering transformations $\mathbb{Z}_p$-conservative. Then $g: Z \to L$ has a maximal $(\mathcal{A}, \mathcal{B})$ tower lifting, which we denote $g_1: Z \to L_1$, so we have a commutative diagram

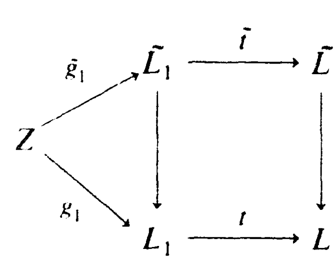


with $t: L_1 \to L$ an $(\mathcal{A}, \mathcal{B})$-tower. We assume all the spaces are pointed and maps preserve the base point.

Lemma. *L_1 is a finite complex and $\pi_1(L_1)$ is p-perfect.*

Proof. Since the combinatorial map g_1 has an image in the CW category, we could replace L_1 by the finite complex $g_1(Z)$ to get an extension of the tower t and lift of g. Thus $L_1 = g_1(Z)$ is finite. If $\pi_1(L_1)$ were not p-perfect, then $\Pi_p(\pi_1(L_1)) \neq \pi_1(L_1)$ (see 1.5 for notation). Let $G = \pi_1(L_1)/\Pi_p(\pi_1(L_1))$. By 2.3, G is $\mathbb{Z}_p$-conservative, so defines a proper cover $L_2 \xrightarrow{p} L_1$ in the class $\mathcal{B}$. Since Z is 1-connected, the map $Z \xrightarrow{g_1} L_1$ lifts through p to give $g_2: Z \to L_2$ such that $p \circ g_2 = g_1$. This contradicts the maximality of g_1. Thus $\pi_1(L_1)$ is p-perfect and the lemma is established.

6.4. Preserving the notation of 6.3, we let $K_1 = L_1^{(2)}$ and observe that $L_1 \setminus K_1$ is a union of n and $(n+1)$ cells. Pick a base point in the cover pair $(\tilde{L}_1, \tilde{K}_1)$ and denote $G_1 = \pi_1(L_1)$. There is a lift $\tilde{g}_1: Z \to \tilde{L}_1$ of g_1,



and a lift $\tilde{t}:\tilde{L}_1\to\tilde{L}$ of t determined by the base points. One has $t\circ g_1=g$ and $\tilde{t}\circ\tilde{g}_1=\tilde{g}$, again by the fact base points correspond. Let $x_1=(\tilde{g}_1)_*(z)\in H_{n+1}(\tilde{L}_1,\tilde{K}_1;\mathbb{Z}_p)$. We see that $\tilde{t}_*(x_1)=x\in H_{n+1}(\tilde{L},\tilde{K};\mathbb{Z}_p)$. If we knew that $H_{n+1}(L_1,K_1;\mathbb{Z}_p)=0$, then we could apply Lemma 6.2 to $\tilde{t}$. Since $G_1=\pi_1(L_1)$ is p-perfect, this would complete the proof of Theorem 4.4. Thus the following result will finish the proof of 4.4:

6.5. Lemma. *Suppose $L_1\xrightarrow{t}L$ is an $(\mathscr{A},\mathscr{B})$ tower with $\mathscr{A}$ and $\mathscr{B}$ as above, where L is a CW complex of dimension $\le n+1$ and $H_{n+1}(L;\mathbb{Z}_p)=0$. Then $H_{n+1}(L_1;\mathbb{Z}_p)=0$.*

Proof. By an induction it suffices to establish this when $t\in\mathscr{A}$ or $t\in\mathscr{B}$. In the latter case, $L_1\xrightarrow{t}L$ is a regular connected cover with group G $\mathbb{Z}_p$-conservative. But $C_.(L_1;\mathbb{Z}_p)$ is a free fg $\mathbb{Z}_pG$ complex and $H_{n+1}(L_1,\mathbb{Z}_p)=\operatorname{Ker} d_{n+1}:C_{n+1}(L_1,\mathbb{Z}_p)\to C_n(L_1;\mathbb{Z}_p)$. This result follows from the definition of $\mathbb{Z}_p$-conservative group G.

If $L_1\to L$ is the inclusion of a finite subcomplex, since there are no cells in dimension $n+2$, we get $H_{n+2}(L,L_1;\mathbb{Z}_p)=0$. The exact sequence

$$H_{n+2}(L,L_1;\mathbb{Z}_p)\to H_{n+1}(L_1;\mathbb{Z}_p)\to H_{n+1}(L;\mathbb{Z}_p)$$

together with $H_{n+1}(L;\mathbb{Z}_p)=0$ gives the result. The proof of 6.5 is complete.

7. Equations in groups

Theorem. 7.1. *Let (L,K) be a pointed connected pair of CW complexes and assume $L\setminus K$ consists of only a finite number of 1- and 2-cells. If $H_2(L,K;\mathbb{Z}_p)=0$, then any element $x\in\operatorname{Ker}(\pi_1(K)\to\pi_1(L))=N$ is contained in an fg p-perfect subgroup of $\pi_1(K)$.*

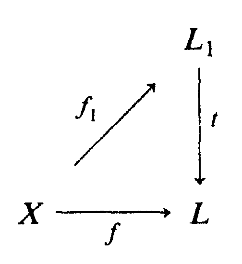
Proof. By [6], there is a combinatorial map $f:X\to L$ of a finite planar 1-connected 2 complex X such that

(1) $f(X^{(1)})\subset K$ and

(2) The image of

$$[S^1,X^{(1)}]\xrightarrow{(f\mid X^{(1)})_*}[S^1,K]$$

contains the conjugacy class of x. By Lemma 5.4, there is a maximal $(\mathscr{A},\mathscr{B})$-tower lifting $f_1:X\to L_1$,



of f, $t\circ f_1=f$, t an $(\mathscr{A},\mathscr{B})$-tower, where $\mathscr{A}$ is the class of inclusions of finite connected subcomplexes and $\mathscr{B}$ is the class of regular (connected) covers with group of covering transformations $\mathbb{Z}_p$-conservative. The argument of 6.3 shows L_1 is finite and $\pi_1(L_1)$ is fg and p-perfect. An argument analogous to 6.5, using the exact sequence of a triple with K_1 = pull back of K in L_1,

$$\begin{array}{ccc} K_1 & \longrightarrow & L_1 \\ \Big\downarrow & \square & \Big\downarrow t \\ K & \longrightarrow & L \end{array}$$

shows $H_2(L_1, K_1; \mathbb{Z}_p)=0$. One deduces that $H_1(K_1;\mathbb{Z}_p)=0$. Now K_1 is in general disconnected, but sits in the connected complex L_1. We may join up the connected components of K_1 by arcs in $L_1^{(1)}$ in such a way that for the resulting finite complex M, $H_1(M,\mathbb{Z}_p)=0$. Denote ϱ the restriction of t to M, $\varrho: M\to K$. Now choose a base point in M, so $\pi_1(M)$ is p-perfect, so $\varrho_*\pi_1(M)$ is an fg p-perfect subgroup of $\pi_1(K)$. But $f_1: X\to L_1$ is such that $f_1(X^{(1)})\subset K_1\subset M$. It follows from the fact that $(f\,|\,X^{(1)})_*([S^1, X^{(1)}])$ contains the conjugacy class of x that there is an element $\xi\in\pi_1(M)$ such that

(1) the conjugacy class of ξ is in $(f_1\,|\,X^{(1)})_*[S^1, X^{(1)}]\subseteq\pi_1(M)$,

(2) $\xi\to 1$ under $\pi_1(M)\to\pi_1(L_1)$

(3) $\varrho_*(\xi)$ is conjugate to x in $\pi_1(K)$.

To see (2), e.g., consider the commutative diagram

$$\begin{array}{ccccc} X^{(1)} & & \subset & & X \\ \Big\downarrow & & & & \Big\downarrow f_1 \\ & \nearrow & M & \searrow & \\ K_1 & & \longrightarrow & & L_1 \end{array}$$

and recall X is 1-connected. Thus $\varrho_*(\xi)\in N=\operatorname{Ker}\pi_1(K)\to\pi_1(L)$. Since $\varrho_*(\xi)$ is conjugate to x, and $\varrho_*(\xi)$ is contained in the fg p-perfect subgroup $\varrho_*(\pi_1(M))$ of $\pi_1(K)$, it follows that x is contained in an fg p-perfect subgroup of $\pi_1 K$. This establishes Theorem 7.1.

7.2. Corollary. $N\le A_p(\pi_1(K))$.

Proof. Let $x\in N$. By the theorem there is an fg p-perfect subgroup H of $G=\pi_1(K)$ with $x\in H$. Now $G/A_p(G)$ is p-conservative by 2.4. Thus $G/A_p(G)$ is l.p.i. by Theorem 4.4. Thus the image of H in $G/A_p(G)$ is trivial, i.e. $H\le A_p(G)$. Hence $x\in A_p(G)$. Since this is true for all $x\in N$, we deduce $N\le A_p(\pi_1(K))$.

7.3. We can rephrase these results in group theoretic language. Let G be a group and $G\langle t_1,\dots,t_n\rangle = G * \langle t_1\rangle * \cdots * \langle t_n\rangle$ where each t_i is an (independent) infinite cycle. Let $w_1,\dots,w_m \in G\langle t_1,\dots,t_n\rangle$ and define the matrix $M=(e_j(w_i))$ whose i-th row is the vector of exponent sums of variables t_j in w_i ($e_j : G\langle t_1,\dots,t_n\rangle \to \mathbb{Z}$ is the homomorphism sending G to 0, and t_i to δ_{ij}, Kronecker's delta). The matrix M is independent (or '∞-independent') if the rows are linearly independent and *p-independent* if the reduction (mod p) of M has rows independent over $\mathbb{Z}_p$ (naturally, p independence implies independence).

Let $H=G\langle t_1,\dots,t_n\rangle/P$ where P is the normal closure of $w_1,\dots,w_m$, and let $\phi : G \to H$ be the induced map. Let $N=\operatorname{Ker}\phi$. Howie's conjecture states that $N=\{1\}$ if the rows of $M=(e_j(w_i))$ are independent.

7.4. Preserving the notations of 7.3, let K be a pointed connected 2-complex with $\pi_1(K)=G$. Let L be obtained from K following the presentation for H. Thus $L=K\vee S^1_{t_1}\vee\cdots\vee S^1_{t_n}\vee e^2_1\vee\cdots\vee e^2_m$ where $S^1_{t_i}$ is attached trivially at the base point of K and e^2_i is attached to $K\vee\bigvee_{i=1}^n S^1_{t_i}$ via $w_i\in\pi_1(K\vee\bigvee_{i=1}^n S^1_{t_i})$. The van Kampen theorem [9] implies $\pi_1(L)=H$ and inclusion $K\to L$ induces $\phi : G\to H$.

7.5. Lemma. *The rows of $M=(e_j(w_i))$ are p independent iff $H_2(L,K;\mathbb{Z}_p)=0$ ($p\le\infty$).*

Proof. The chain differential

$$d_2 : C_2(L,K;\mathbb{Z}_p)\to C_1(L,K;\mathbb{Z}_p)$$

is exactly the matrix M when the groups are given their natural bases of cells. M is p-independent iff $\ker d_2=0$ iff $H_2(L,K;\mathbb{Z}_p)=0$.

7.6. Theorem. *Let G be a group and let $H=G\langle t_1,\dots,t_n\rangle/P$, where P is normally generated by $w_1,\dots,w_m$. Let $N=\operatorname{Ker}(G\to H)$. If $M=(e_j(w_i))$ is p-independent, then $N\le A_p(G)$.*

Proof. Let K and L be constructed as in 7.4. Since M is p-independent, $H_2(L,K;\mathbb{Z}_p)=0$ by 7.5. Corollary 7.2 shows that $N\le A_p(G)$. This completes the proof.

7.7. Corollary. *With the notation of 7.6, if G is locally p-indicable, $N=\{1\}$.*

Proof. If G is l.p.i., G is p-conservative by 4.4. Thus $A_p(G)=\{1\}$, so $N=\{1\}$.

Theorem 7.1 and Corollary 7.7 are due to Howie in the case $p=\infty$. The proof of 7.1 incorporates those mostly notational changes in Howie's argument needed to handle $p<\infty$.

Let us deduce one more application to the 'classical' Kervaire conjecture, one variable and exponent sum 1 [11].

7.8. Theorem. *Let G be a group and $w \in G\langle t\rangle$ with exponent sum of t in w equal to* 1. *Let $H = G\langle t\rangle/P$ where P is the normal closure of w and let N be the kernel of the composite $G \to G\langle t\rangle \twoheadrightarrow H$. Then $N \leq \bigcap_{p\leq\infty} A_p(G)$.*

7.9. Example. Let $G = S_3$. Then $A_2(G) = A_3$ and $A_p(G) = G$, $p \neq 2$. Thus $\bigcap_{p\leq\infty} A_p(G) = A_3$. On the other hand, $N = \{1\}$ by the theorem of Gerstenhaber and Rothaus [4].

7.10. I cannot resist closing with a remark about Higman's group,

$$G = \langle x_0, x_1, x_2, x_3 \,|\, x_i = [x_{i+1}, x_i], i \text{ (mod 4)}\rangle$$

(see [12, p. 18). Since G is perfect, $A_p(G) = G$. Also G has no proper subgroups of finite index. Thus G is not locally residually finite. It would appear natural to look for a counterexample to Howie's conjecture [6] in G if one were disinclined to believe it, or prove it for G if one wanted to produced good evidence for it.

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